ON THE REDUCTION OF CIRCUIT EQUATIONS FOR NUMERICAL INTEGRATION

by

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THESIS

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ABSTRACT

The reduction of circuit equations to normal form for numerical integration is considered for the general circuit where excess reactive elements, all types of linearly dependent sources, and nonlinear dissipative and reactive elements are present. For the linear circuit, necessary and sufficient conditions for the existence of numerical solutions are considered and stated. For the nonlinear circuit, reduction to normal form is not always possible. Numerical solution is shown to be simplified if certain a priori conditions are satisfied in formulating the original circuit equations. A new systematic reduction procedure is presented for obtaining the normal form equations. This procedure is also extended to a new procedure for obtaining transfer and immittance functions of the linear circuit from a proper tree formulation.

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I. INTRODUCTION

The formulation of circuit equations for their numerical solution is of current interest. The initial contribution was made by Bashkow, who formulated the concept of the proper tree.[1,2,3] This was followed by the work of Bryant, and Kuh and Rohrer who refined the techniques. The object of their procedure, which was based primarily upon topology, is to obtain the circuit equations in matrix form so that one has a first-order differential equation. The approach uses the state-variable technique, which has dominated the literature. However, it should be noted that other formulations of circuit equations, such as the TRAC Program, do not use state variables.

Once the circuit equations have been formulated by the proper-tree concept, they have the following form

$$A\dot{x} + Bw + Cz = Dx + Ew + F\dot{z} + Gu$$

where x is the vector of voltages and currents associated with independent energy-storing elements, w is the vector of currents and voltages associated with dissipative elements, z is the vector of currents and voltages associated with excess reactive elements, u is the vector of independent sources, and the capital letters are coefficient matrices.

For numerical solution it is necessary to alter the form to the normal form[1], $\dot{x} = Ax + Bu$ for the linear case and $\dot{x} = f(x,u,t)$ for the non-linear case, so that numerical integration can be applied directly.

It has been shown that the circuit matrix equation, based on the formulation of the proper tree, gives the relation of the currents and voltages of each circuit element to the currents and voltages of all other circuit elements. Of this set of circuit variables, currents and voltages of the circuit elements, there is a minimal set called states that are required to describe the circuit. If those variables which do not belong to the minimal set can be eliminated the resulting matrix equation will be of normal form, or an equation that can be readily converted to normal form.

Chapter II is a study of the circuit that contains linear circuit elements and linearly dependent sources. The formation of the proper tree is discussed. A new systematic technique for matrix reduction that eliminates the unwanted circuit variables is developed. The circuit that contains no dependent sources has a matrix equation that always reduces to normal form and reveals that the number of states is equal to the number of independent energy-storing elements in the circuit. The introduction of dependent sources alters the coefficient matrices of the circuit equation. These alterations make the reduction process conditional and these conditions are discussed. In addition, it is found that the introduction of dependent sources may alter the number of states required to describe the circuit. Those reactive variables that are not destined to become states are designated as surplus. It is shown that the surplus reactive variables can always be eliminated from the equations.

In Chapter III the reduction techniques developed in Chapter II are utilized in establishing a new procedure for deriving driving-point impedance/admittance functions and transfer functions of the linear circuit. This procedure enables a transfer immittance to be calculated directly from the proper tree formulation. Derivatives are replaced by their frequency-domain equivalents.

In Chapter IV nonlinear dissipative elements are included with the linear circuit elements. It is shown that if normal form state equations are to be derived the currents and voltages of the nonlinear dissipative elements must be restricted in their relationship to the voltages and currents of other circuit elements. This relationship may be controlled by a priori circuit modifications that increase the number of states.

Chapter V extends the discussion to nonlinear reactive elements. Two new theorems relating to the conditions for the elimination of dissipative variables are stated. It is also revealed that the surplus reactive variables of the general nonlinear circuit must be associated linear elements and that these currents/voltages must not be related to the currents/voltages of nonlinear elements if normal form equations are desired.

II. REDUCTION OF LINEAR CIRCUIT EQUATIONS TO NORMAL FORM

State equations are often used in circuit analysis. Normal form state equations, $\dot{x} = f(x,u,t)$, may be obtained from the matrix equation of the circuit's currents and voltages.

A. FORMATION OF THE CIRCUIT MATRIX EQUATION

In order to derive circuit equations in a consistent manner a proper tree is used. Any tree for the circuit may be used to obtain the circuit equations. However, if other than the proper tree is used, additional steps of transposing the variables to the proper side of the equation are necessary in the reduction process. The proper tree is drawn as follows. All voltage sources are designated as tree branches and all current sources are designated as links. This precludes a loop of voltage sources, which necessarily must be constrained so that the loop can be eliminated, or a cut-set of current sources, which must be constrained so that the cut-set can be eliminated, as illustrated in Figure 1. All capacitive elements, less the minimum required to prevent the formation of a loop, are designated as tree branches (C); those exluded become links (elastance, S). All inductive elements, less the minimum required to prevent the formation of a cut-set, are designated as links (L); those excluded become tree branches (inverse inductance, Γ). Resistive elements which do not complete loops are designated as tree branches (conductance, G); the excess become links (resistance, R). Thus the order of precedence for drawing tree branches is (1) voltage sources, (2) capacitive elements, (3) dissipative elements, and (4) inductive elements.

It is desirable to give consideration to elements whose currents and voltages are used as controlling terms for dependent sources. When a source is voltage dependent, the element across which the controlling voltage is developed should be a tree branch if this constraint does not upset the precedence for forming a proper tree. When a source is current controlled, the element through which the controlling current flows should be a link. Should these arrangements not be possible, the controlling element's branch current may be written in terms of the link currents of elements forming a cut-set with the controlling element. Similarly, a link element's voltage may be written in terms of the branch voltages of elements forming a loop with the controlling element. Should the network be too complex to readily perform these substitutions and arrangements, this special consideration should be disregarded. Most circuits have very few dependent sources in comparison to other circuit elements. This consideration causes the dependency constants to appear in the circuit equations with the constants dictated by the network topology.

Kirchhoff's voltage law is written for each fundamental loop defined by links L, R, and S. Kirchhoff's current law is written for each fundamental cut-set defined by tree branches C, G, and Γ. All voltage sources appear in loop equations and all current sources appear in cut-set equations. Figures 2 and 3 illustrate the conceivable loops and cut-sets.

A loop defined by an L may contain dependent voltage sources, independent voltage sources, C's, G's and Γ's. A loop defined by an R may contain all of the foregoing with the exception of Γ's. A loop defined by an S does not contain G's and Γ's. A cut-set defined by a C may contain L's, R's, S's, and current sources. Cut-sets defined by G's do not contain S's and cut-sets defined by Γ's do not contain S's and R's. By using a proper tree other loops and cut-sets are prohibited.

In the equations which follow, F (with appropriate subscripts)[1] relates link and tree branch voltages and link and tree branch voltages and link and tree branch currents according to the topology of the circuit. A prime designates the transpose of a matrix. e_L, e_R, and e_S represent independent voltage sources. j_Γ, j_G, and j_C represent independent current sources. r and μ (with appropriate subscripts) are submatrices of the proportionality constants of linearly dependent voltage sources. h and g (with appropriate subscripts) are the proportionality constants of the linearly dependent current sources. h and μ are dimensionless whereas r and g have dimensions of ohms and mhos respectively. In the matrix equation which follows, fundamental loop voltage equations and fundamental cut-set current equations are written alternately as indicated. Thus, the circuit equations become (1).

$$
\left(\begin{bmatrix}
\mu_{LL} & r_{LC} & \mu_{LR} & r_{LG} & \mu_{LS} & r_{L\Gamma} \\
g_{CL} & h_{CC} & g_{CR} & h_{GC} & g_{CS} & h_{C\Gamma} \\
\mu_{RL} & r_{RC} & \mu_{RR} & r_{RG} & \mu_{RS} & r_{R\Gamma} \\
g_{GL} & h_{GC} & g_{GR} & h_{GG} & g_{GS} & h_{G\Gamma} \\
\mu_{SL} & r_{SC} & \mu_{SR} & r_{SG} & \mu_{SS} & r_{S\Gamma} \\
g_{\Gamma L} & h_{\Gamma C} & g_{\Gamma R} & h_{\Gamma G} & g_{\Gamma S} & h_{\Gamma\Gamma}
\end{bmatrix} + I\right)
\begin{bmatrix}
L\dot{I}_L \\ CV_C \\ RI_R \\ GV_G \\ V_S \\ I_\Gamma
\end{bmatrix}
=
\begin{bmatrix}
0 & F_{LC} & 0 & F_{LG} & 0 & F_{L\Gamma} \\
-F'_{LC} & 0 & -F'_{RC} & 0 & -F'_{SC} & 0 \\
0 & F_{RC} & 0 & F_{RG} & 0 & 0 \\
-F'_{LG} & 0 & -F'_{RG} & 0 & 0 & 0 \\
0 & F_{SC} & 0 & 0 & 0 & 0 \\
-F'_{L\Gamma} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
r_{LL} & \mu_{LC} & r_{LR} & \mu_{LG} & r_{LS} & \mu_{L\Gamma} \\
h_{CL} & g_{CC} & h_{CR} & g_{CG} & h_{CS} & g_{C\Gamma} \\
r_{RL} & \mu_{RC} & r_{RR} & \mu_{RG} & r_{RS} & \mu_{R\Gamma} \\
h_{GL} & g_{GC} & h_{GR} & g_{GG} & h_{GS} & g_{G\Gamma} \\
r_{SL} & \mu_{SC} & r_{SR} & \mu_{SG} & r_{SS} & \mu_{S\Gamma} \\
h_{\Gamma L} & g_{\Gamma C} & h_{\Gamma R} & g_{\Gamma G} & h_{\Gamma S} & g_{\Gamma\Gamma}
\end{bmatrix}
\begin{bmatrix}
I_L \\ V_C \\ I_R \\ V_G \\ S^{-1}\dot{V}_S \\ \Gamma^{-1} I_\Gamma
\end{bmatrix}
+
\begin{bmatrix}
e_L \\ j_C \\ e_R \\ j_G \\ e_S \\ j_\Gamma
\end{bmatrix}
\qquad (1)
$$

B. THE REDUCTION PROCESS

The reduction process includes the elimination of the variables associated with the dissipative elements, the transposition of the variables associated with the excess reactive elements, and the elimination of the coefficient matrix from the left side of the reduced equations. The order of the first two steps is unimportant.

Partitioning (1) as indicated and lettering

$$[x] = \begin{bmatrix} I_L \\ V_C \end{bmatrix} \quad ; \quad [w] = \begin{bmatrix} I_R \\ V_G \end{bmatrix} \quad ; \quad [z] = \begin{bmatrix} V_S \\ I_T \end{bmatrix} \tag{2}$$

enables (1) to be written as (3).

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \dot{x} \\ w \\ z \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{23} \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{z} \end{bmatrix} + \begin{bmatrix} e_x \\ e_w \\ e_z \end{bmatrix} \tag{3}$$

In (3), x designates the state variables, w designates the "unwanted" dissipative variables, and z designates the excess reactive element variables. Appendix A relates the submatrices of (3) to the submatrices of (1).

1. Elimination of Dissipative Variables

The following rearrangement of (3) shows the relation of the variables, w, to the other variables and independent sources.

$$\begin{bmatrix} (P_{12} - Q_{12}) \\ (P_{22} - Q_{22}) \\ (P_{32} - Q_{32}) \end{bmatrix} [w] = \begin{bmatrix} Q_{11} & Q_{13} \\ Q_{21} & Q_{23} \\ Q_{31} & Q_{33} \end{bmatrix} \begin{bmatrix} x \\ \dot{z} \end{bmatrix} - \begin{bmatrix} P_{11} & P_{13} \\ P_{21} & P_{23} \\ P_{31} & P_{33} \end{bmatrix} \begin{bmatrix} \dot{x} \\ z \end{bmatrix} + \begin{bmatrix} e_x \\ e_w \\ e_z \end{bmatrix} \tag{4}$$

To have a determinate solution for the variables, w, the rank of the matrix multiplying w in (4) must be equal to the number of dissipative elements in the circuit. This matrix has dimensions n by m. Where n is the total number of circuit elements and m is the total number of dissipative elements in the circuit. Because of the dependent sources, whose proportionality constants may be any real number, there exists the possibility that the matrix multiplying w in (4) will have rank less than m. In this event the solution for the variables, w, will be indeterminate.

Let the rank of the matrix multiplying w in (4) be m-i because of row dependence. Rearranging the rows of (4) to group the rows such that the first tier is dependent upon the second tier leads to the following.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w_{(i)} \\ w_{(m-i)} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} x \\ \dot{z} \end{bmatrix} - \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \dot{x} \\ z \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{4a}$$

M_{22} is a square matrix of rank m-i and dimensions m-i by m-i. The enclosed subscripts denote the dimensions of the w vector partitions. Since the first tier of M is a linear combination of the second tier, there must exist some matrix K such that $K \begin{bmatrix} M_{22} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \end{bmatrix}$. Multiplying the second tier of (4a) by -K and adding the results to the first tier yields

$$\begin{bmatrix} 0 & 0 \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w_{(i)} \\ w_{(m-i)} \end{bmatrix} = \begin{bmatrix} (N_{11}-KN_{21}) & (N_{12}-KN_{22}) \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} x \\ \dot{z} \end{bmatrix} - \begin{bmatrix} (T_{11}-KT_{21}) & (T_{12}-KT_{22}) \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \dot{x} \\ z \end{bmatrix} + \begin{bmatrix} e_1-Ke_2 \\ e_2 \end{bmatrix} \tag{4b}$$

The second tier of (4b) yields the indeterminate solution for w.

$$w_{(m-i)} = M_{22}^{-1}(N_{21}x + N_{22}\dot{z} - T_{21}\dot{x} - T_{22}z - M_{21}w_{(i)} + e_2) \tag{4c}$$

Recalling that there were n elements in the circuit, m of which were dissipative, it can be seen that the first tier of (4b) contains n-m+i equations relating to n-m reactive variables. There is no guarantee that solutions obtained from a set of n-m of these equations would be consistent with solutions obtained from a different set. Assuming that the state equations obtained from the n-m+i equations were consistent, the substitution of these states and their derivatives into (4c) does not yield a determinate solution for the dissipative variables. Therefore, it is necessary that the coefficient matrix of the w variables in (4) have rank equal to the number of dissipative elements in the circuit if the w variables are to be eliminated.

The submatrix $\left[P_{22}-Q_{22}\right]$ has dimensions m by m. If the rank is m, then the solution for w is found in the second tier of (4).

$$w = (P_{22}-Q_{22})^{-1} (Q_{21}x + Q_{23}\dot{z} - P_{21}\dot{x} - P_{23}z + e_w) \tag{5}$$

Substituting (5) into the first and third tiers of (4) yields

$$\left[\begin{array}{c|c} \epsilon & \zeta \\ \hline \eta & \lambda \end{array}\right] \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}\right] \left[\begin{array}{c} x \\ \hline \dot{z} \end{array}\right] + \left[\begin{array}{c} V_1 \\ \hline V_2 \end{array}\right] \tag{6}$$

where

$$\alpha = Q_{11} + (Q_{12}-P_{12})(P_{22}-Q_{22})^{-1} Q_{21} \tag{7a}$$

$$\beta = Q_{13} + (Q_{12}-P_{12})(P_{22}-Q_{22})^{-1} Q_{23} \tag{7b}$$

$$\gamma = Q_{31} + (Q_{32}-P_{32})(P_{22}-Q_{22})^{-1} Q_{21} \tag{7c}$$

$$\delta = Q_{33} + (Q_{32}-P_{32})(P_{22}-Q_{22})^{-1} Q_{23} \tag{7d}$$

$$\epsilon = P_{11} + (Q_{12}-P_{12})(P_{22}-Q_{22})^{-1} P_{21} \tag{7e}$$

$$\zeta = P_{13} + (Q_{12}-P_{12})(P_{22}-Q_{22})^{-1} P_{23} \tag{7f}$$

$$\eta = P_{31} + (Q_{32} - P_{32})\,(P_{22} - Q_{22})^{-1}\, P_{21} \tag{7g}$$

$$\lambda = P_{33} + (Q_{32} - P_{32})\,(P_{22} - Q_{22})^{-1}\, P_{23} \tag{7h}$$

$$V_1 = e_x + (Q_{12} - P_{12})\,(P_{22} - Q_{22})^{-1}\, e_w \tag{7i}$$

$$V_2 = e_z + (Q_{32} - P_{32})\,(P_{22} - Q_{22})^{-1}\, e_w \tag{7j}$$

In the event that $(P_{22}-Q_{22})$ is singular, the matrix inverse used in (5) would not exist. However, the n rows of (4) may be rearrange to provide a square matrix of rank m. This matrix may then be used in lieu of $(P_{22}-Q_{22})$ in an equation similar to (5) yielding results similar to (6). Failure to achieve a suitable arrangement of the rows (4) leads to an indeterminate solution for the variables, w. Since these variables are resistor currents and voltages and must remain finite, indeterminate solutions are not anticipated in circuit analysis problems.

2. Transposition of Excess Reactive Variables

To achieve the normal form equations from (6), z and $\dot{z}$ must be transposed. This transposition yields

$$\left[\begin{array}{c|c} \epsilon & \beta \\ \hline \eta & \delta \end{array}\right] \left[\begin{array}{c} \dot{x} \\ \hline \dot{z} \end{array}\right] = \left[\begin{array}{c|c} \alpha & \zeta \\ \hline \gamma & \lambda \end{array}\right] \left[\begin{array}{c} x \\ z \end{array}\right] + \left[\begin{array}{c} V_1 \\ \hline -V_2 \end{array}\right] \tag{8}$$

The variables, z, are currents and voltages of reactive elements that were deemed excess in the formation of the proper tree. In a circuit free of dependent sources these variables would never become state variables. Because of the dependent sources (8) shows that this distinction can no longer be made. Thus, if

$$y = \begin{bmatrix} x \\ z \end{bmatrix}; \quad A = \begin{bmatrix} \epsilon & -\beta \\ -\eta & \delta \end{bmatrix}; \quad B = \begin{bmatrix} \alpha & \zeta \\ \gamma & \lambda \end{bmatrix}; \quad U = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix} \tag{9}$$

(8) becomes

$$A\dot{y} = By + U \tag{10}$$

Further reduction depends upon the rank of the matrix A. If A is nonsingular, then (10) yields the normal form

$$\dot{y} = A^{-1}By + A^{-1}U \tag{11}$$

If the matrix A is singular, (11) is not valid and additional steps in the reduction process are required.

3. Elimination of Surplus Reactive Variables

The matrix A in (10) can be singular because of linear dependence between the rows or between the columns or both. If the rows of (10) are regrouped forming one tier which consists of the independent rows of the matrix A and a second tier with the remaining rows and these tiers are partitioned to form square submatrices along the principal diagonals of A and B, the following results.

$$\begin{bmatrix} A_{ii} & A_{id} \\ A_{di} & A_{dd} \end{bmatrix} \begin{bmatrix} \dot{y}_i \\ \dot{y}_d \end{bmatrix} = \begin{bmatrix} B_{ii} & B_{id} \\ B_{di} & B_{dd} \end{bmatrix} \begin{bmatrix} y_i \\ y_d \end{bmatrix} + \begin{bmatrix} U_i \\ U_d \end{bmatrix} \tag{12}$$

In (12) the subscripts i and d denote independence and dependence respectively.

Since the second tier of the matrix A in (12) is a linear combination of the first tier, there exists some matrix K such that $\begin{bmatrix} A_{di} & A_{dd} \end{bmatrix} = K \begin{bmatrix} A_{ii} & A_{id} \end{bmatrix}$. Multiplying the first tier by -K and adding the results to the second tier yields

$$\begin{bmatrix} A_{ii} & A_{id} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_i \\ \dot{y}_d \end{bmatrix} = \begin{bmatrix} B_{ii} & B_{id} \\ (B_{di}-KB_{ii}) & (B_{dd}-KB_{id}) \end{bmatrix} \begin{bmatrix} y_i \\ y_d \end{bmatrix} + \begin{bmatrix} U_i \\ U_d \end{bmatrix} \tag{13}$$

Assuming that $(B_{dd}-KB_{id})$ is nonsingular, the second tier of (13) yields

$$y_d = (B_{dd}-KB_{id})^{-1} \quad (KB_{ii}-B_{di})y_i + KU_i - U_d \tag{14}$$

Substituting (14) and the assoicated derivative equations into the first tier of (13) yields

$$A^*\dot{y}_i = B^*y_i + C^*U_i + D^*U_d + E^*\dot{U}_i + F^*\dot{U}_d \tag{15}$$

where

$$A^* = A_{ii} + A_{id}(B_{dd}-KB_{id})^{-1} \quad (KB_{ii}-B_{di}) \tag{16a}$$

$$B^* = B_{ii} + B_{id}(B_{dd}-KB_{id})^{-1} \quad (KB_{ii}-B_{di}) \tag{16b}$$

$$C^* = B_{id}(B_{dd}-KB_{id})^{-1}K + I \tag{16c}$$

$$D^* = -B_{id}(B_{dd}-KB_{id})^{-1} \tag{16d}$$

$$E^* = -A_{id}(B_{dd}-KB_{id})^{-1}K \tag{16e}$$

$$F^* = A_{id}(B_{dd}-KB_{id})^{-1} \tag{16f}$$

If A* is singular because of row dependence the process used to derive (15) from (10) is repeated. Each repetition of this process may yield higher-order derivatives of the source terms. If A* is singular because of column dependence a process similar to that used to derive (15) is employed.

Assume that the matrix A in (10) is singular because of dependence between the columns. Regrouping and partitioning (10) accordingly yield an equation identical to (12) where

$$\begin{bmatrix} A_{id} \\ A_{dd} \end{bmatrix} = \begin{bmatrix} A_{ii} \\ A_{di} \end{bmatrix} \begin{bmatrix} K \end{bmatrix} \tag{17}$$

Multiplying the first tier of (12), partitioned according to column dependence, by A_{ii}^{-1} yields

$$\begin{bmatrix} I & K \\ A_{di} & A_{dd} \end{bmatrix} \begin{bmatrix} \dot{y}_i \\ \dot{y}_d \end{bmatrix} = \begin{bmatrix} A_{ii}^{-1}B_{ii} & A_{ii}^{-1}B_{id} \\ B_{di} & B_{dd} \end{bmatrix} \begin{bmatrix} y_i \\ y_d \end{bmatrix} + \begin{bmatrix} A_{ii}^{-1}U_i \\ U_d \end{bmatrix} \tag{18}$$

Multiplying the first tier of (18) by $-A_{di}$ and adding the results to the second tier yields (19).

$$\begin{bmatrix} I & K \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_i \\ \dot{y}_d \end{bmatrix} = \begin{bmatrix} A_{ii}^{-1}B_{ii} & A_{ii}^{-1}B_{id} \\ (B_{di}-A_{di}A_{ii}^{-1}B_{ii}) & (B_{dd}-A_{di}A_{ii}^{-1}B_{id}) \end{bmatrix} \begin{bmatrix} y_i \\ y_d \end{bmatrix} + \begin{bmatrix} A_{ii}^{-1}U_i \\ U_d-A_{di}A_{ii}^{-1}U_i \end{bmatrix} \tag{19}$$

Substituting the solution for y_d from the second tier of (19) and the associated derivative equations into the first tier yields

$$*A\dot{y}_i = *By_i + *CU_i + *DU_d + *E\dot{U}_i + *F\dot{U}_d \tag{20}$$

where

$$*A = I - K(B_{dd}-A_{di}A_{ii}^{-1}B_{id})^{-1} (B_{di}-A_{di}A_{ii}^{-1}B_{ii}) \tag{21a}$$

$$*B = A_{ii}^{-1}B_{ii}-A_{ii}^{-1}B_{id}(B_{dd}-A_{di}A_{ii}^{-1}B_{id})^{-1} (B_{di}-A_{di}A_{ii}^{-1}B_{ii}) \tag{21b}$$

$$*C = A_{ii}^{-1} + A_{ii}^{-1}B_{id}(B_{dd}-A_{di}A_{ii}^{-1}B_{id})^{-1} A_{di}A_{ii}^{-1} \tag{21c}$$

$$*D = -A_{ii}^{-1}B_{id}(B_{dd}-A_{di}A_{ii}^{-1}B_{id})^{-1} \tag{21d}$$

$$*E = -K(B_{dd}-A_{di}A_{ii}^{-1}B_{id})^{-1}A_{di}A_{ii}^{-1} \tag{21e}$$

$$*F = K(B_{dd}-A_{di}A_{ii}^{-1}B_{id})^{-1} \tag{21f}$$

If the matrix *A is singular and has row dependence, the process beginning with (12) is repeated. If there is only column dependence, the latter part of the process is repeated.

In the foregoing it was assumed that the matrices $(B_{dd}-KB_{id})$, used in (13) through (16), and $(B_{dd}-A_{di}A_{ii}^{-1}B_{id})$, used in (19) through (21), were nonsingular. If these matrices are singular, there will exist some rearrangement of the columns of (13) and (19) that will provide nonsingular matrices that can be used in place of $(B_{dd}-KB_{id})$ and $(B_{dd}-A_{di}A_{ii}^{-1}B_{id})$. This statement is proved by contradictions that show that the rank of the second-tier submatrix of y coefficients in (13) or (19) is equal to the number of variables to be eliminated. First, it is shown that there are no dependent rows; secondly, it is shown that the number of independent columns is equal to or greater than the number of variables to be eliminated.

Let the following represent the arrangement of (13) or (19) in which the independent rows of the second tier of the y coefficient matrix have been separated from the dependent rows, the dependent rows being placed in a third tier.

$$\begin{bmatrix} A_{ii} & A_{id} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_i \\ \dot{y}_d \end{bmatrix} = \begin{bmatrix} B_{ii} & B_{id} \\ B^*_{di} & B^*_{dd} \\ B''_{di} & B''_{dd} \end{bmatrix} \begin{bmatrix} y_i \\ y_d \end{bmatrix} + \begin{bmatrix} U_i \\ U^*_d \\ U''_d \end{bmatrix} \tag{22}$$

Since $\begin{bmatrix} B''_{di} & B''_{dd} \end{bmatrix} = K \begin{bmatrix} B^*_{di} & B^*_{dd} \end{bmatrix}$, K being the matrix that shows the linear combination of the second tier of (22) that results in the third tier, multiplying the second tier by -K and adding the results to the third tier yields

$$\begin{bmatrix} A_{ii} & A_{id} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_i \\ \dot{y}_d \end{bmatrix} = \begin{bmatrix} B_{ii} & B_{id} \\ B^*_{di} & B^*_{dd} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_i \\ y_d \end{bmatrix} + \begin{bmatrix} U_i \\ U^*_d \\ U''_d \; -KU^*_d \end{bmatrix} \tag{22a}$$

It can be seen that the third tier of (22a) contains only equations of constraint on independent sources. Since these sources cannot be constrained and remain independent, it must be assumed that the second tiers of the y coefficient matrices in (13) and (19) contain no dependent rows.

Assume that the second tiers of (13) and (19) do not contain a sufficient number of independent columns in the y coefficient matrices to perform the required substitutions. Let the following represent the rearrangement of the second tiers of (13) or (19) which provides one square, nonsingular submatrix, the first tiers not being shown.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} y^* \\ y'' \end{bmatrix} + \begin{bmatrix} U^* \\ U'' \end{bmatrix} \tag{22b}$$

Let C_{11} be the nonsingular, square submatrix. Since the second set of columns of C is a linear combination of the first set, there must exist some matrix K such that

$$\begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \begin{bmatrix} K \end{bmatrix} \tag{22c}$$

Multiplying the first tier of (22b) by C_{11}^{-1} yields

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & K \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} y^* \\ y'' \end{bmatrix} + \begin{bmatrix} C_{11}^{-1}U^* \\ U'' \end{bmatrix} \tag{22d}$$

Multiplying the first tier of (22d) by $-C_{21}$ and adding the results to the second tier yields

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & K \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y^* \\ y'' \end{bmatrix} + \begin{bmatrix} C_{11}^{-1}U^* \\ U''-C_{21}C_{11}^{-1}U^* \end{bmatrix} \tag{22e}$$

Again constraint equations on the independent sources have appeared. Thus, it must be assumed that there are a sufficient number of independent columns in the second tiers of the y coefficient matrices in (13) and (19) to perform the operations required to eliminate the surplus reactive variables. Hence, the rank of the second-tier submatrix of y coefficients in (13) and (19) is equal to the number of variables to be eliminated.

C. CONCLUSIONS

In a circuit without dependent sources the number of states required to describe the circuit is equal to the number of capacitive branches plus the number of inductive links in the proper tree. When the circuit contains dependent sources, the proper tree does not necessarily determine the number of states. It can be seen by comparing (7) to (1) that additional states maybe required if the dependent sources are controlled by currents or voltages of reactive elements made excess by the formation of the proper tree.

Since the dependent source proportionality constants may take on any value, there exists the possibility that fewer states than predicted by the proper tree may be used to describe the circuit. Because of matrix singularities some of the reactive variables become surplus. The removal of these surplus variables introduces derivatives of the independent source terms and reduces the number of states.

D. EXAMPLES

1. Increase in States Due to Dependent Sources

Consider the circuit of Figure 4 which illustrates the effect of dependent sources that requires additional states to describe the circuit. The circuit equations for Figure 4(a) are the following.

$$\begin{bmatrix} L_1 \dot{I}_{L1} \\ L_2 \dot{I}_{L2} \\ C_2 \dot{V}_{C2} \\ C_3 V_{C3} \\ V_{S1} \\ I_{L3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ V_{C2} \\ V_{C3} \\ C_1 \dot{V}_{S1} \\ L_3 \dot{I}_{L3} \end{bmatrix} \qquad (23)$$

With the absence of dissipative elements (22) may be expressed in the form of (6) in which

$$\epsilon = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix} ; \alpha = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} ; \beta = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & L_3 \end{bmatrix} \qquad (23a)$$

$$\lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad \gamma = \begin{bmatrix} 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \end{bmatrix} ; \quad \zeta = \eta = \delta = 0 \tag{23b}$$

Transposing the excess reactive variables, the equations become

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 & -L_3 \\ 0 & L_2 & 0 & 0 & 0 & -L_3 \\ 0 & 0 & C_2 & 0 & -C_1 & 0 \\ 0 & 0 & 0 & C_3 & -C_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I}_{L1} \\ \dot{I}_{L2} \\ \dot{V}_{C2} \\ \dot{V}_{C3} \\ \dot{V}_{S1} \\ \dot{I}_{\Gamma 3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ V_{C2} \\ V_{C3} \\ V_{S1} \\ I_{\Gamma 3} \end{bmatrix} \tag{24}$$

(24) is in the form of (13) and the variables V_{S1} and $I_{\Gamma 3}$ may be eliminated. With the dependent sources present as shown in Figure 4(b).

$$\delta = \begin{bmatrix} r & \mu \\ h & g \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & L_3 \end{bmatrix} \tag{25}$$

and the equations in the form of (24) become

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 & -L_3 \\ 0 & L_2 & 0 & 0 & 0 & -L_3 \\ 0 & 0 & C_2 & 0 & -C_1 & 0 \\ 0 & 0 & 0 & C_3 & -C_1 & 0 \\ 0 & 0 & 0 & 0 & rC_1 & \mu L_3 \\ 0 & 0 & 0 & 0 & hC_1 & gL_3 \end{bmatrix} \begin{bmatrix} \dot{I}_{L1} \\ \dot{I}_{L2} \\ \dot{V}_{C2} \\ \dot{V}_{C3} \\ \dot{V}_{S1} \\ \dot{I}_{\Gamma 3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ V_{C2} \\ V_{C3} \\ V_{S1} \\ I_{\Gamma 3} \end{bmatrix} \tag{26}$$

If δ is nonsingular, none of the variables can be eliminated. If δ is singular with rank of one, either V_{S1} or $I_{\Gamma 3}$ may be eliminated. Hence it can be seen that the presence of the dependent sources of Figure 4(b) caused an increase in the number of states required to describe the circuit.

2. Decrease in States Due to Dependent Sources

Consider the circuit of Figure 5 which illustrates the effect of dependent sources that reduces the number of states by causing matrix singularities. The circuit equations for Figure 5 are the following.

$$\begin{bmatrix} L\dot{I}_L \\ C\dot{V}_C \\ R_1 I_{R1} \\ G_2 V_{G2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -R_1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & R_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_L \\ V_C \\ I_{R1} \\ V_{G2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} V \end{bmatrix} \tag{27}$$

The equations of (27) are converted to the form of (3). Since there are no excess reactive elements $P_{31} = Q_{31} = P_{32} = Q_{32} = P_{33} = Q_{33} = P_{13} = Q_{13} = P_{23} = Q_{23} = 0$.

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & G_2 \end{bmatrix} \begin{bmatrix} \dot{I}_L \\ \dot{V}_C \\ I_{R1} \\ V_{G2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -R_1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & R_1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_L \\ V_C \\ I_{R1} \\ V_{G2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} V \end{bmatrix} \tag{28}$$

From (28) it can be seen that $(P_{22}-Q_{22})$ is singular and can not be used in the solution for the dissipative variables. Noting that $P_{12}-Q_{12})$ is nonsingular, the first two rows of (28) are used to solve for these variables.

$$\begin{bmatrix} I_{R1} \\ V_{G2} \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & R_1 \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{I}_L \\ \dot{V}_C \end{bmatrix} = \begin{bmatrix} 0 & C \\ -L & -CR_1 \end{bmatrix} \begin{bmatrix} \dot{I}_L \\ \dot{V}_C \end{bmatrix} \tag{29}$$

Substituting (29) into the second tier of (28) yields

$$\begin{bmatrix} 0 & 0 \\ G_2 L & G_2 CR_1 \end{bmatrix} \begin{bmatrix} \dot{I}_L \\ \dot{V}_C \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} V \end{bmatrix} \tag{30}$$

(30) is in the form of (13). From the first row it can it can be seen that $V_C = V$. Thus substituting $\dot{V}$ for $\dot{V}_C$ in the second row of (30) yields the final result.

$$G_2 L\dot{I}_L = -I_L \quad -G_2 C G_1 \dot{V} \tag{31}$$

From (31) it can be seen that V_C is not needed to describe the circuit of Figure 5. Had the proportionality constant r been any other value, V_C would have been a state. Thus it can be seen that dependent sources may reduce the number of states required to describe the circuit.

III. LINEAR CIRCUIT EQUATIONS REDUCED TO IMMITTANCE FORMS

Linear circuit equations derived by the proper-tree concept can be reduced to impedance, admittance, and transfer functions if the circuit equations are augmented by output equations. When dealing with transfer/immittance functions the Laplace operator with zero initial conditions must be used in lieu of derivative notation. If the circuit equations are $\dot{x} = Ax + Bu$ and the output equations are $y = Cx + Du$, then x may be found from the circuit equations and substituted into the output equations to yield $Y(s) = [C(sI-A)^{-1}B + D]\,U(s)$. This is an extended determination of a transfer immittance which reduces to the usual one when there is a single input and a single output. This procedure requires one to obtain the state equations and output equations and from them derive equations that yield the desired functions. The procedure developed here enables one to derive the desired functions directly from (1) by reduction techniques.

A. DERIVATION OF DRIVING POINT ADMITTANCE/IMPEDANCE

If a network is driven at one set of terminals by a voltage source and a proper tree is drawn of the network, only one fundamental cut-set will be described by the source. The sum of the currents of the link elements forming the cut-set with the source is the current passing through the source. Thus, the circuit equations may be augmented with an output equation which expresses the source current in terms of link currents. Solving for the source current in terms of the source voltage results in an equation of the following form.

$$I(s) = Y(s)V(s) \tag{32}$$

I(s) is the source current, Y(s) is by definition the input admittance function, and V(s) is the source voltage function. If V(s) is unity (32) becomes

$$I(s) = Y(s) \tag{33}$$

If the network is driven by a unit current source, only one fundamental loop will be defined by the source when the proper tree is drawn. The voltage developed by the source is the sum of the voltages of the branch elements forming the loop with the source. Thus, the circuit equations may be augmented with an output equation which expresses the source voltage in terms of the branch voltages. Solving for the source voltage in terms of the unit current source results in an equation of the following form.

$$V(s) = Z(s) \tag{34}$$

V(s) is the source voltage function, and Z(s) is the input impedance function.

Writing the circuit equations in the form of (1), with $L\dot{I}_L$ replaced by sLI_L, etc., and performing the substitutions given by (2) yields an equation similar to (3), as follows.

$$\begin{bmatrix} sP_{11} & P_{12} & P_{13} \\ sP_{21} & P_{22} & P_{23} \\ sP_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} x \\ w \\ z \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & sQ_{13} \\ Q_{21} & Q_{22} & sQ_{23} \\ Q_{31} & Q_{32} & sQ_{33} \end{bmatrix} \begin{bmatrix} x \\ w \\ z \end{bmatrix} + \begin{bmatrix} e_x \\ e_w \\ e_z \end{bmatrix} \tag{35}$$

Augmenting (35) with the output equation yields

$$\left[\begin{array}{ccc|c} sP_{11} & P_{12} & P_{13} & 0 \\ sP_{21} & P_{22} & P_{23} & 0 \\ sP_{31} & P_{32} & P_{33} & 0 \\ \hline sP^*_{ox} & F^*_{ow} & F^*_{oz} & I \end{array}\right] \left[\begin{array}{c} x \\ w \\ z \\ \hline \theta \end{array}\right] = \left[\begin{array}{ccc} Q_{11} & Q_{12} & sQ_{13} \\ Q_{21} & Q_{22} & sQ_{23} \\ Q_{31} & Q_{32} & sQ_{33} \\ \hline F^{**}_{ox} & F^{**}_{ow} & sF^{**}_{oz} \end{array}\right] \begin{bmatrix} x \\ w \\ z \end{bmatrix} + \left[\begin{array}{c} e_x \\ e_w \\ e_z \\ \hline 0 \end{array}\right] \tag{36}$$

where θ is the output current/voltage to be determined and F_{ox}, F_{ow}, and F_{oz} are terms which relate the output to the circuit voltages and currents. The coupling terms, F* and F**, may include dependent source constants as well as topological factors.

In the reduction process of Chapter II, to obtain the normal form equations, all dissipative currents and voltages were eliminated. The elimination process caused the dissipative parameters to appear in the coefficient matrices. The output equation identifies the currents and voltages of interest and to obtain an immittance the proper-tree equations should be reduced by eliminating those currents and voltages that do not appear in the output equation. Thus, after writing the equations in the form of (36), the unwanted variables are eliminated. The reduction process for these variables is identical to the process used to eliminate the dissipative variables in the derivation of normal form equations. (see Appendix C)

The elimination of the unwanted variables, those for which the total output coupling (F** -F*) is zero, reduces the dimensions of the matrix equation. In addition, the reduction may cause the coefficients of the reduced equation to become polynomials or ratios of polynomials of the variables. The reduced equation is of the following form.

$$\begin{bmatrix} P_r & 0 \\ - & - \\ F^*_r & I \end{bmatrix} \begin{bmatrix} y \\ - \\ \theta \end{bmatrix} = \begin{bmatrix} Q_r \\ - \\ F^{**}_r \end{bmatrix} [y] + \begin{bmatrix} e_y \\ - \\ 0 \end{bmatrix} \tag{36a}$$

where the subscript r denotes the alteration of the coefficient matrices and y denotes the circuit variables that were not eliminated. Any of the elements of P_r, Q_r, F^*_r, and F^{**}_r may now be functions of the variable s. However, this notation has been omitted for simplicity in writing.

Solving for y in the first tier of (36a) yields

$$[y] = \left[P_r - Q_r\right]^{-1} \left[e_y\right] \tag{36b}$$

Substituting this solution for y into the second tier of (36a) and solving for θ yields

$$\theta = \left[F^{**}_r - F^*_r\right] \left[P_r - Q_r\right]^{-1} e_y \tag{37}$$

If θ is the current of a unit voltage source, then the operations indicated by (37) yield an admittance function. If θ is the voltage developed by a unit current source, (37) yields an impedance function.

B. DERIVATION OF TRANSFER FUNCTIONS

Network transfer functions may also be derived from the foregoing equations by letting θ become a voltage or current at terminals not directly connected to the source. Care must be exercized when the output is to be an open-circuit voltage as in the case of a transfer impedance function. There may be elements of the circuit that are in series with the open-circuit terminals that do not affect the open-circuit voltage. These elements should be excluded from the proper tree and the resulting circuit equations.

If the network is excited by a unit current source and θ is the open-circuit voltage at another set of terminals, then (37) yields a transfer impedance. If θ is a current, the right side of (37) becomes a current gain function. If the network is excited by a unit voltage source and θ is a current through another set of terminals, the right side of (37) becomes a transfer admittance function. If θ is a voltage, a voltage gain function is derived.

By letting θ become an output vector, both input and transfer functions may be derived simultaneously. Thus (37) may be extended to multiple-port circuits with multiple excitations by identifying the sources. Since the output equations determine the unwanted variables, it may be necessary to perform the elimination process in stages to obtain various input and transfer functions.

C. SUMMARY

The steps to the derivation of driving-point and transfer functions are summarized as follows:

1. determine the sources needed to derive the desired functions
2. eliminate the variables that appear in none of the output equations
3. perform the operations indicated by (37) on those output equations where the output is coupled to all the remaining variables including those variables that are not coupled but not effective.

D. EXAMPLES

Consider the network shown in Figure 6a. Assume that the input admittance, Y_{11}, for port 1-1' is desired. Excite the circuit with a unit voltage source at port 1-1'. The proper tree is shown in Figure 6b. The augmented circuit equations are

$$\begin{bmatrix} sL_1 I_{L1} \\ sL_2 I_{L2} \\ sC_1 V_{C1} \\ sC_2 V_{C2} \\ R_1 I_{R1} \\ R_3 I_{R3} \\ G_2 V_{G2} \\ \hline Y_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ V_{C1} \\ V_{C2} \\ I_{R1} \\ I_{R3} \\ V_{G2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{bmatrix} \tag{38}$$

The last equation is the output equation which shows the fundamental cut-set relation of the source current and I_{L1}. The remaining circuit variables are to be eliminated. Elimination of the dissipative variables from (38) yields

$$\begin{bmatrix} sL_1 I_{L1} \\ sL_2 I_{L2} \\ sC_1 C_V \\ sC_2 V_{C2} \\ \hline Y_{11} \end{bmatrix} = \begin{bmatrix} -R_2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -G_1 & 0 \\ 1 & -1 & 0 & -G_3 \\ \hline 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ V_{C1} \\ V_{C2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{bmatrix} \tag{39}$$

Elimination of the capacitive variables from (39) yields

$$\begin{bmatrix} sL_1I_{L1} \\ sL_2I_{L2} \\ \hline Y_{11} \end{bmatrix} = \begin{bmatrix} -(R_2 + \frac{1}{sC_1 + G_1} + \frac{1}{sC_2 + G_3}) & (\frac{1}{sC_2 + G_3}) \\ (\frac{1}{sC_2 + G_3}) & -(\frac{1}{sC_2 + G_3}) \\ \hline 1 & 0 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

Eliminating I_{L2} and performing the operations indicated by (37) yields

$$Y_{11} = \frac{as^3 + bs^2 + cs + d}{es^4 + fs^3 + gs^2 + hs + i} \tag{41}$$

where

$$a = L_2C_1C_2 \tag{41a}$$

$$b = G_1L_2C_2 + G_3L_2C_1 \tag{41b}$$

$$c = G_1G_3L_2 + C_1 \tag{41c}$$

$$d = G_1 \tag{41d}$$

$$e = L_1L_2C_1C_2 \tag{41e}$$

$$f = G_1L_1L_2C_2 + R_2L_2C_1C_2 + G_3L_1L_2C_1 \tag{41f}$$

$$g = G_1G_3L_1L_2 + G_1R_2L_2C_2 + R_2G_3L_2C_1 + L_1C_1 + L_2C_1 + L_2C_2 \tag{41g}$$

$$h = G_1R_2G_3L_2 + G_1L_1 + G_1L_2 + R_2C_1 + G_3L_2 \tag{41h}$$

$$i = G_1R_2 + 1 \tag{41i}$$

Assume that the two-port $\mathcal{Z}$ parameter matrix for the network of Figure 6a is desired. The proper tree with both ports being excited by current sources is shown in Figure 6c. The augmented circuit equations are

$$\begin{bmatrix} sL_2 I_{L2} \\ sC_1 V_{C1} \\ sC_2 V_{C2} \\ R_1 I_{R1} \\ R_3 I_{R3} \\ G_2 V_{G2} \\ I_{\Gamma 1} \\ \hline e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{L2} \\ V_{C1} \\ V_{C2} \\ I_{R1} \\ I_{R3} \\ V_{G2} \\ sL_1 I_{\Gamma 1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{42}$$

From the output equations it can be seen that I_{L2}, I_{R1}, and I_{R3} should be eliminated. The result of their elimination and the relocation of sL_1 is

$$\begin{bmatrix} sC_1 V_{C1} \\ sC_2 V_{C2} \\ G_2 V_{G2} \\ I_{\Gamma 1} \\ \hline e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -G_1 & 0 & 0 & 0 \\ 0 & -(G_3 + \frac{1}{sL_2}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & sL_1 \\ 0 & 1 & 0 & sL_1 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{G2} \\ I_{\Gamma 1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{43}$$

Performing the steps necessary to obtain the form of (37) yields

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & sL_1 \\ 0 & sL_1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{sC_1 + G_1} & 0 & 0 & 0 \\ 0 & \dfrac{sL_2}{L_2C_2s^2 + L_2G_3 + 1} & 0 & 0 \\ 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{44}$$

Completion of the multiplication indicated by (44) reveals the desired parameters. The results are

$$z_{11} = \frac{1}{C_1s + G_1} + \frac{sL_2}{L_2C_2s^2 + L_2G_3s + 1} + R_2 + L_1s \tag{45}$$

$$z_{12} = z_{21} = z_{22} = \frac{sL_2}{L_2C_2s^2 + L_2G_3s + 1} + L_1s \tag{46}$$

The solutions have been left in a form that allows one to check their accuracy by inspection.

The hybrid parameters could be derived by exciting the circuit at one port with a voltage source and a current source at the other port.

IV. REDUCTION TO NORMAL FORM WITH NONLINEAR DISSIPATORS

Because of the nonlinear current-to-voltage relations of semiconductors, the numerical solutions for circuit containing these devices may become complex if the state variable approach is used. The circuit can be represented by a set of normal-form equations that are augmented by equations which also must be solved to obtain the nonlinear currents and voltages that appear in the normal-form equations. The general form of these equations are

$$\begin{aligned} \dot{x} &= A_1 x + B_1 u + C f_1(v) + D f_2(i) \\ v &= A_2 x + B_2 u + f_3(i,v) \\ i &= A_3 x + B_3 u + f_4(i,v) \end{aligned} \tag{47}$$

where the f'x are nonlinear functions and v and i are the voltages and currents of the semiconductors. If f_3 and f_4 are zero, then (47) reduces to one nonlinear differential matrix equation which may be integrated directly.

Diodes either by themselves, or as part of the Ebers-Moll[4] equivalent for the transistor may be considered in terms of their current/voltage characteristics. For the solid-state diode it is possible to write

$$i_d = I_o(e^{kv_d} - 1)$$

and

$$v_d = \frac{1}{k} \ln\left(\frac{i_d + I_o}{I_o}\right)$$

The exact nonlinear relation is unimportant for the present consideration so that the form

$$i_d = f(v_d) \tag{48a}$$

and the inverse form

$$v_d = g(i_d) \tag{48b}$$

are used here.

Werther and Parker[5] and Calahan[6] consider the diode to be a current-dependent voltage source or a voltage-dependent current source where the controlling voltage or current is written in terms of the voltages and currents of other circuit elements as in (47). This treatment requires the solution of network equations for the determination of the controlling voltages and currents. If the both the current function, (48a), and the voltage function, (48b), are single valued (the tunnel diode is exluded), then the nonlinear dissipative element may be treated as the linear dissipative elements are treated when writing circuit equations. That is, the device may be used as either a branch element or a link element when forming a proper tree.

A. TOPOLOGICAL CONSIDERATIONS

The nonlinear elements are given precedence between capacitive elements and linear dissipative elements when establishing the branches of the proper tree. When writing a voltage equation for a loop defined by a nonlinear, dissipative link element, a term on the left side of the equation such as RI_R is replaced by the voltage v_d. The term corresponding to

I_R of the right side of the equation is $f(v_d)$. Similarly, when writing a current equation of a cut-set defined by a nonlinear, dissipative element, the term on the left side of the equation is i_d and the term on the right side is $g(i_d)$. The matrix equation for the circuit will be similar to that of (1).

B. ELIMINATION OF DISSIPATIVE VARIABLES

If state equations are to be obtained the dissipative variables must be eliminated. While the linear dissipative variables may readily be eliminated, the elimination of the nonlinear variables depends upon the relationship of the currents/voltages of the nonlinear elements to the voltages and currents of other circuit elements. This may be seen by the following discussion.

1. Nonlinear Variables As General Functions of Other Variables

Assume that the nonlinear voltages/currents are functions of all other voltages and currents. After writing the circuit equations and completing the substitutions given by (2) the equations become

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix} \begin{bmatrix} x \\ z \\ w \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} \end{bmatrix} \begin{bmatrix} x \\ \dot{z} \\ w \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} e_x \\ e_z \\ e_w \\ e_d \\ j_d \end{bmatrix} \tag{49}$$

where x, z, and w are given by (2) and P_{ij} and Q_{ij} (i, j, = 1, 2, 3) are given by Appendix A. P_{ij} and Q_{ij} (i = 4, 5; j = 1, 2, 3, 4, 5 and j = 4, 5;

$i = 1,2,3,4,5$) are matrices which relate the voltages of nonlinear dissipative links to other voltages in the fundamental loops which are defined by nonlinear elements, and the currents of nonlinear branches to the currents of other elements which are in the cut-sets that are defined by nonlinear, dissipative branch elements.

The first two tiers and corresponding columns of (49) may be combined by transposing z and $\dot{z}$. Thus, by forming y from x and z as was done in (9), (49) becomes the following.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \dot{y} \\ w \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} \begin{bmatrix} y \\ w \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} e_y \\ e_w \\ e_d \\ j_d \end{bmatrix} \tag{50}$$

As was discussed in Chapter II, the w variables may be eliminated if the rank of the combined coefficient matrices for these variables is equal to the number of these variables. Their elimination reduces (50) to the following.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} \dot{y} \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} y \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{51}$$

where, if $(A_{22}-B_{22})$ is nonsingular,

$$K_{11} = A_{11} - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} A_{21} \tag{52a}$$

$$K_{12} = A_{13} - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} A_{23} \tag{52b}$$

$$K_{13} = A_{14} - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} A_{24} \tag{52c}$$

$$K_{21} = A_{31} - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} A_{21} \tag{52d}$$

$$K_{22} = A_{33} - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} A_{23} \tag{52e}$$

$$K_{23} = A_{34} - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} A_{24} \tag{52f}$$

$$K_{31} = A_{41} - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} A_{21} \tag{52g}$$

$$K_{32} = A_{43} - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} A_{23} \tag{52h}$$

$$K_{33} = A_{44} - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} A_{24} \tag{52i}$$

$$M_{11} = B_{11} - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} B_{21} \tag{52j}$$

$$M_{12} = B_{13} - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} B_{23} \tag{52k}$$

$$M_{13} = B_{14} - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} B_{24} \tag{52l}$$

$$M_{21} = B_{31} - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} B_{21} \tag{52m}$$

$$M_{22} = B_{33} - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} B_{23} \tag{52n}$$

$$M_{23} = B_{34} - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} B_{24} \tag{52o}$$

$$M_{31} = B_{41} - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} B_{21} \tag{52p}$$

$$M_{32} = B_{43} - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} B_{23} \tag{52q}$$

$$M_{33} = B_{44} - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} B_{24} \tag{52r}$$

$$V_1 = e_y - \left[A_{12} - B_{12}\right]\left[A_{22} - B_{22}\right]^{-1} e_w \tag{52s}$$

$$V_2 = e_d - \left[A_{32} - B_{32}\right]\left[A_{22} - B_{22}\right]^{-1} e_w \tag{52t}$$

$$V_3 = j_d - \left[A_{42} - B_{42}\right]\left[A_{22} - B_{22}\right]^{-1} e_w \tag{52u}$$

The multiplication of (51) by $[K]^{-1}$ yields (47). Further reduction depends upon the elements of M_{22}, M_{23}, M_{32}, and M_{33}. This may be seen by writing the second two tiers of (51) as follows.

$$\begin{bmatrix} v_d \\ i_d \end{bmatrix} = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix}^{-1} \left(- \begin{bmatrix} K_{21} \\ K_{31} \end{bmatrix} [\dot{y}] + \begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} [y] + \begin{bmatrix} V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} f(v_d) \\ g(i_d) \end{bmatrix} \right) \quad (53)$$

Equation (53) suggest several undesirable solutions for the nonlinear currents and voltages. Transcendental solutions may be required. If these are not required, solutions which reveal that a nonlinear current/voltage is a nonlinear function of other currents/voltages may result. The substitution of the latter solutions into the first tier of (51) yields functions of functions. If $M_{22}=M_{23}=M_{32}=M_{33} = 0$, these undesirable solutions cannot exist. By comparing (52) with (50), it can be seen that the desired conditions prevail if $B_{23}=B_{24}=B_{32}=B_{33}=B_{34}=B_{42}=B_{43}=B_{44} = 0$. This implies that the fundamental loops defined by nonlinear elements must not contain other dissipative elements, either linear or nonlinear. Similarly, cut-sets defined by nonlinear elements must not contain other dissipative elements. These restrictions must also apply to dependent sources which are controlled by the currents/voltages of dissipative elements. Thus, if desirable equations are to result, the nonlinear, dissipative variables must be restricted to being functions of reactive variables only.

2. Nonlinear Variables As Functions of Reactive Variables

If the currents and voltages of nonlinear elements are not allowed to be controlling terms for dependent sources and these currents/voltages can only be related to capacitor voltages or inductor currents, then (50) becomes the following.

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{y} \\ w \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & 0 & 0 \\ B_{31} & 0 & 0 & 0 \\ B_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ w \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} e_y \\ e_w \\ e_d \\ j_d \end{bmatrix} \qquad (54)$$

The eliminations of the w variables yields the following.

$$\begin{bmatrix} K_{11} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{y} \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & 0 & 0 \\ M_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \qquad (55)$$

The elimination of v_d and i_d from (55) yields

$$K_{11}\dot{y} = M_{11}y + V_1 + M_{12}f(M_{21}y + V_2) + M_{13}g(M_{31}y + V_3) \qquad (56)$$

If K_{11} in (56) is nonsingular, then (56) readily yields the normal form equations. If K_{11} is singular some of the variables in y must be eliminated. The elimination of the surplus reactive variables may be possible but very difficult. Assume that the left side of (56) is zero and that one of the variables y is to be written in terms of the other variables and sources. It can be seen that the solution of a transcendental equation may be involved. To investigate the conditions necessary for reduction when K_{11} is singular, a circuit that is free of dependent sources is considered.

For a circuit that does not contain dependent sources the circuit equations reduce to the following.

$$\begin{bmatrix} \dot{x} \\ z \\ w \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & 0 & 0 & 0 & 0 \\ Q_{31} & 0 & Q_{33} & Q_{34} & Q_{35} \\ Q_{41} & 0 & Q_{43} & 0 & Q_{45} \\ Q_{51} & 0 & Q_{53} & Q_{54} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{z} \\ w \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} e_x \\ e_z \\ e_w \\ e_d \\ j_d \end{bmatrix} \tag{57}$$

The matrices Q_{34}, Q_{35}, Q_{43}, and Q_{53} will be other than zero whenever a fundamental cut-set is formed linear dissipative elements and nonlinear dissipative elements. Likewise, Q_{45} and Q_{54} will be other than zero whenever a fundamental cut-set is formed with nonlinear dissipative elements. The elimination of the linear dissipative variables gives rise to coefficients for $f(v_d)$ and $g(i_d)$ in the reduced equations causing the solutions for v_d and i_d to become transcendental. If i_d is eliminated from (57), then Q_{45} and Q_{54} cause the solution for v_d to become transcendental. Thus, normal-form state equations will not result from the reduction of circuit equations when fundamental cut-sets are formed with nonlinear dissipative elements and other dissipative elements.

If the matrices mentioned in the preceeding paragraph were zero, the nonlinear voltages/currents would be functions of the state variables and sources only and all dissipative variables could be eliminated. The resulting normal form equations would be as follows.

$$\dot{x} = Ax + BU + C\dot{U} + Df(Ex + FU) + Gg(Hx + JU) \tag{58}$$

C. CONCLUSIONS

The circuit equations for circuits containing nonlinear dissipative elements do not readily reduce to normal-form state equations. By a priori circuit modifications of placing an inductor in series with those nonlinear elements which have loop equations formed with variables that are not destined to become state variables and placing a capacitor in parallel with those nonlinear elements which have cut-set equations formed with variables that are not destined to become state variables, the nonlinear elements' voltages and currents are forced to become state variables. While these modifications do give rise to additional states they allow the circuit equations to be reduced to normal-form equations.

D. EXAMPLE

Consider the circuit shown in Figure 7a. The circuit equations are the following.

$$\begin{bmatrix} C\dot{V}_C \\ L\dot{I}_L \\ i_{d1} \\ v_{d2} \end{bmatrix} = \begin{bmatrix} -G_2 & 1 & 0 & 0 \\ -1 & -R_1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_C \\ I_L \\ g(i_{d1}) \\ f(v_{d2}) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \qquad (59)$$

This equation cannot be put in normal form because of the coupling between diodes. If the circuit is modified as in Figure 7b, the circuit equations become

$$\begin{bmatrix} C\dot{V}_C \\ C_1\dot{V}_{C1} \\ L\dot{I}_L \\ v_{d1} \\ v_{d2} \end{bmatrix} = \begin{bmatrix} -G_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ -1 & -1 & -R_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_C \\ V_{C1} \\ I_L \\ f(v_{d1}) \\ f(v_{d2}) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{60}$$

The diodes are now uncoupled and (60) may be placed in normal form.

$$\begin{bmatrix} C\dot{V}_C \\ C_1\dot{V}_{C1} \\ L\dot{I}_L \end{bmatrix} = \begin{bmatrix} -G_2 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -R_1 \end{bmatrix} \begin{bmatrix} V_C \\ V_{C1} \\ I_L \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f(V_{C1}) \\ f(V_{C1}-V_1-V_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ V_1 \end{bmatrix} \tag{61}$$

(61) is obtained by eliminating v_d from (60). Normal-form equations are obtained by multiplying (61) by

$$\begin{bmatrix} C & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & L \end{bmatrix}^{-1} \tag{62}$$

V. REDUCTION TO NORMAL FORM WITH GENERAL NONLINEAR ELEMENTS

When nonlinear inductors and capacitors are present, the circuit equations may be reduced to the following form.

$$\begin{bmatrix} P_{ij} \end{bmatrix} \begin{bmatrix} \dot{x} \\ f_z(z) \\ w \\ v_d \\ i_d \end{bmatrix} = \begin{bmatrix} Q_{ij} \end{bmatrix} \begin{bmatrix} f_x(x) \\ \dot{z} \\ w \\ f(v_d) \\ g(i_d) \end{bmatrix} + \begin{bmatrix} e_x \\ e_z \\ e_w \\ e_d \\ j_d \end{bmatrix} \tag{63}$$

where

$$x = \begin{bmatrix} \phi_L \\ Q_c \end{bmatrix} \; ; \; f_x(x) = \begin{bmatrix} I_L \\ V_c \end{bmatrix} = \begin{bmatrix} f_L(\phi_L) \\ f_c(Q_c) \end{bmatrix} \tag{64a,b}$$

and

$$z = \begin{bmatrix} Q_S \\ \phi_\Gamma \end{bmatrix} \; ; \; f_z(z) = \begin{bmatrix} V_S \\ I_\Gamma \end{bmatrix} = \begin{bmatrix} f_S(Q_S) \\ f_\Gamma(\phi_\Gamma) \end{bmatrix} \tag{65a,b}$$

ϕ_L is the flux associated with inductors and Q_c is the charge associated with capacitors. ϕ_L and Q_c are functionally related to the inductor currents and capacitor voltages as given by (64b). For the linear case $\phi_L = LI_L$ and $Q_c = CV_c$. (65a,b) give the general relationships for excess reactive elements.

If x and z in (49) are replaced by $f_x(x)$ and $f_z(z)$ respectively, the result is (63). Since $\dot{z}$ and $f_z(z)$ may be transposed, it is possible to form

$$\dot{y} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} \quad \text{and} \quad f_y(y) = \begin{bmatrix} f_x(x) \\ f_z(z) \end{bmatrix} . \tag{66a,b}$$

If $f_y(y)$ is used in place of y in the equations that begin with (50), the arguments for reduction with nonlinear inductors and capacitors would be obtained. Thus, if the dissipative variables are to be eliminated, the conditions for the elimination of these variables from (49) must also apply to (63).

The conditions for the elimination of the dissipative variables are stated in the following theorems.

<u>Theorem 1.</u> If a coefficient submatrix for the linear dissipative variables in the equations which relate these variables to the other circuit variables and sources has rank equal to the number of linear dissipative elements, the linear dissipative variables may be eliminated and normal-form equations may be derived. (For proof see Chapter IIB1)

<u>Theorem 2.</u> If each of the voltages or currents of nonlinear dissipative elements, nonlinear dissipative variables, are linearly related to only independent sources and the reactive voltages and currents that are functions of variables that are destined to become state variables, then the nonlinear dissipative variables may be eliminated and normal form equations may be derived. (For proof see Chapter IVB2)

Having eliminated the dissipative variables from (63), the resulting equations are as follows.

$$A\dot{y} = Bf_y(y) + U + Cf_d(Df_y(y) + U) + Eg_d(Hf_y(y) + U) \tag{67}$$

If A in (67) is singular there exist surplus states to be eliminated. Since these states do not appear in functions f_d and g_d which result from the elimination of the nonlinear dissipative variables, an equation of the following form must be solved to eliminate the surplus state functions $f_y(y)$ and the surplus state derivatives.

$$0 = \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} f_y(y)_i \\ f_y(y)_s \end{bmatrix} + U \tag{68}$$

where the subscripts i and s denote the state functions to be retained and the surplus state functions respectively. While the surplus state functions $f_y(y)_s$ may be written in terms of the other state functions and sources, the solution for the surplus state derivatives involves partial derivatives of both the surplus functions and the functions to be retained.

$$-\frac{\partial f_y(y)_s}{\partial y_s} \dot{y}_s = N^{-1} M \frac{\partial f_y(y)_i}{\partial y_i} \dot{y}_i + N^{-1} \dot{U} \tag{69}$$

From (69) it can be seen that if the surplus states are to be eliminated the surplus state functions must be linear and must be related to retained functions that are also linear. Otherwise, the partial derivatives must be entered into the coefficient matrix of the retained derivatives and the resulting equations become nonlinear differential equations that cannot be placed in normal form.

APPENDIX A

General Submatrix Relations to Circuit Equation Submatrices

$$P_{11} = \begin{bmatrix} \mu_{LL} & r_{LC} \\ g_{CL} & h_{CC} \end{bmatrix} + I \; ; \quad Q_{11} = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & F_{LC} \\ -F'_{LC} & 0 \end{bmatrix} + \begin{bmatrix} r_{LL} & \mu_{LC} \\ h_{CL} & g_{CC} \end{bmatrix} \right)$$

$$P_{12} = \begin{bmatrix} \mu_{LR} & h_{CG} \\ g_{CR} & h_{CG} \end{bmatrix} \; ; \quad Q_{12} = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & F_{LG} \\ -F'_{RC} & 0 \end{bmatrix} + \begin{bmatrix} r_{LR} & \mu_{LG} \\ h_{CR} & g_{CG} \end{bmatrix} \right)$$

$$P_{13} = \begin{bmatrix} \mu_{LS} & r_{L\Gamma} \\ g_{CS} & h_{C\Gamma} \end{bmatrix} \; ; \quad Q_{13} = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & F_{L\Gamma} \\ -F'_{SC} & 0 \end{bmatrix} + \begin{bmatrix} r_{LS} & \mu_{R\Gamma} \\ h_{CS} & g_{C\Gamma} \end{bmatrix} \right) \begin{bmatrix} S^{-1} & 0 \\ 0 & \Gamma \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} \mu_{RL} & r_{RC} \\ g_{GL} & h_{GC} \end{bmatrix} \; ; \quad Q_{21} = \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & F_{RC} \\ -F'_{LG} & 0 \end{bmatrix} + \begin{bmatrix} r_{RL} & \mu_{RC} \\ h_{GL} & g_{GC} \end{bmatrix} \right)$$

$$P_{22} = \begin{bmatrix} \mu_{RR} & r_{RG} \\ g_{GR} & h_{GG} \end{bmatrix} + I \; ; \quad Q_{22} = \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & F_{RG} \\ -F'_{RG} & 0 \end{bmatrix} + \begin{bmatrix} r_{RR} & \mu_{RG} \\ h_{GR} & g_{GG} \end{bmatrix} \right)$$

$$P_{23} = \begin{bmatrix} \mu_{RS} & r_{R\Gamma} \\ g_{GS} & h_{G\Gamma} \end{bmatrix} \; ; \quad Q_{23} = \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix}^{-1} \begin{bmatrix} r_{RS} & \mu_{R\Gamma} \\ h_{GS} & g_{G\Gamma} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & \Gamma^{-1} \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} \mu_{SL} & r_{SC} \\ g_{\Gamma L} & g_{\Gamma C} \end{bmatrix} \; ; \quad Q_{31} = \begin{bmatrix} 0 & F_{SC} \\ -F'_{L\Gamma} & 0 \end{bmatrix} + \begin{bmatrix} r_{SL} & \mu_{SC} \\ h_{\Gamma L} & g_{\Gamma C} \end{bmatrix}$$

$$P_{32} = \begin{bmatrix} \mu_{SR} & r_{SG} \\ g_{\Gamma R} & h_{\Gamma G} \end{bmatrix} \; ; \quad Q_{32} = \begin{bmatrix} r_{SR} & \mu_{SG} \\ h_{\Gamma R} & g_{\Gamma G} \end{bmatrix}$$

$$P_{33} = \begin{bmatrix} \mu_{SS} & r_{S\Gamma} \\ g_{\Gamma S} & h_{\Gamma\Gamma} \end{bmatrix} + I \; ; \quad Q_{33} = \begin{bmatrix} r_{SS} & \mu_{S\Gamma} \\ h_{\Gamma S} & g_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & \Gamma^{-1} \end{bmatrix}$$

$$e_x = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}^{-1} \begin{bmatrix} e_L \\ j_C \end{bmatrix} \; ; \quad e_w = \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix}^{-1} \begin{bmatrix} e_R \\ j_G \end{bmatrix} \; ; \quad e_z = \begin{bmatrix} e_S \\ j_\Gamma \end{bmatrix}$$

$$x = \begin{bmatrix} I_L \\ V_C \end{bmatrix} ; \quad w = \begin{bmatrix} I_R \\ V_G \end{bmatrix} ; \quad z = \begin{bmatrix} V_S \\ I_\Gamma \end{bmatrix}$$

APPENDIX A

General Submatrix Relations to Circuit Equation Submatrices

$$P_{11} = \begin{bmatrix} (I + \mu_{LL}) & r_{LC} \\ g_{CL} & (I + h_{CC}) \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \; ; \quad P_{12} = \begin{bmatrix} \mu_{LR} & r_{LG} \\ g_{CR} & h_{CG} \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix} \; ; \quad P_{13} = \begin{bmatrix} \mu_{LS} & r_{L\Gamma} \\ g_{CS} & h_{C\Gamma} \end{bmatrix} \tag{4a}$$

$$P_{21} = \begin{bmatrix} \mu_{RL} & r_{RC} \\ g_{GL} & h_{GC} \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \; ; \quad P_{22} = \begin{bmatrix} (1 + \mu_{RR}) & r_{RG} \\ g_{GR} & (1 + h_{GG}) \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix} \; ; \quad P_{23} = \begin{bmatrix} \mu_{RS} & r_{R\Gamma} \\ g_{GS} & h_{G\Gamma} \end{bmatrix} \tag{4b}$$

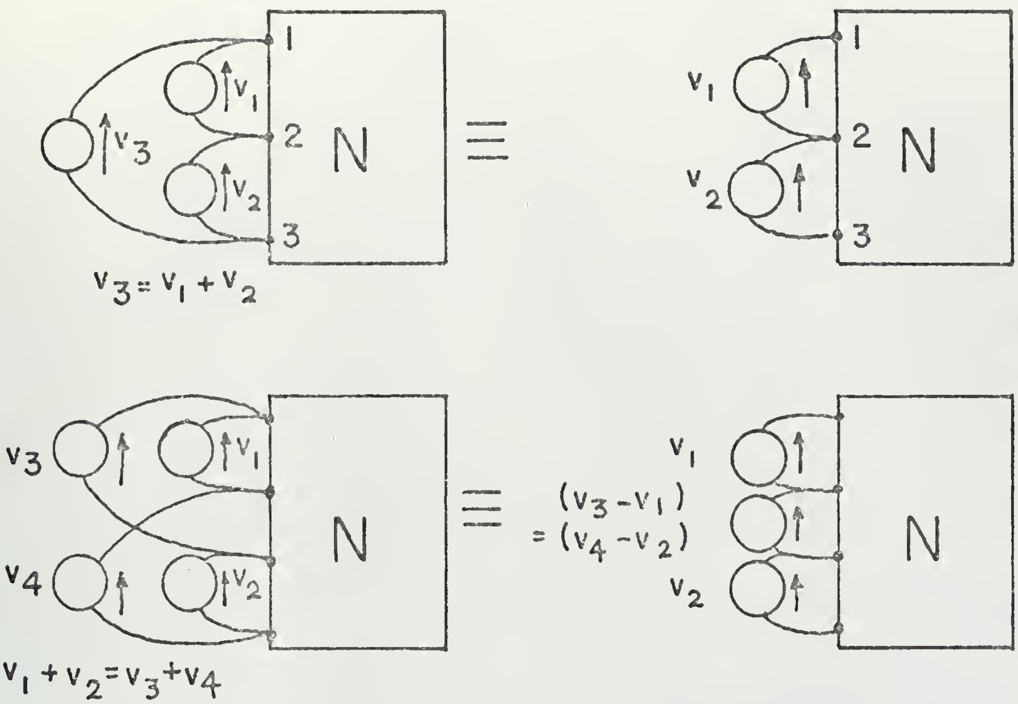
$$P_{31} = \begin{bmatrix} \mu_{SL} & r_{SC} \\ g_{\Gamma L} & h_{\Gamma C} \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \; ; \quad P_{32} = \begin{bmatrix} \mu_{SR} & r_{SG} \\ g_{\Gamma R} & h_{\Gamma G} \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & G \end{bmatrix} \; ; \quad P_{33} = \begin{bmatrix} (1 + \mu_{SS}) & r_{S\Gamma} \\ g_{\Gamma S} & (1 + h_{\Gamma\Gamma}) \end{bmatrix} \tag{4c}$$

$$Q_{11} = \begin{bmatrix} r_{LL} & (\mu_{LC} - F_{LC}) \\ (h_{CL} + F'_{LC}) & g_{CC} \end{bmatrix} \; ; \quad Q_{12} = \begin{bmatrix} r_{LR} & (\mu_{LG} - F_{LG}) \\ (h_{CR} + F'_{RC}) & g_{CG} \end{bmatrix} \; ; \quad Q_{13} = \begin{bmatrix} r_{LS} & (\mu_{L\Gamma} - F_{L\Gamma}) \\ (h_{CS} + F'_{SC}) & g_{C\Gamma} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \tag{4d}$$

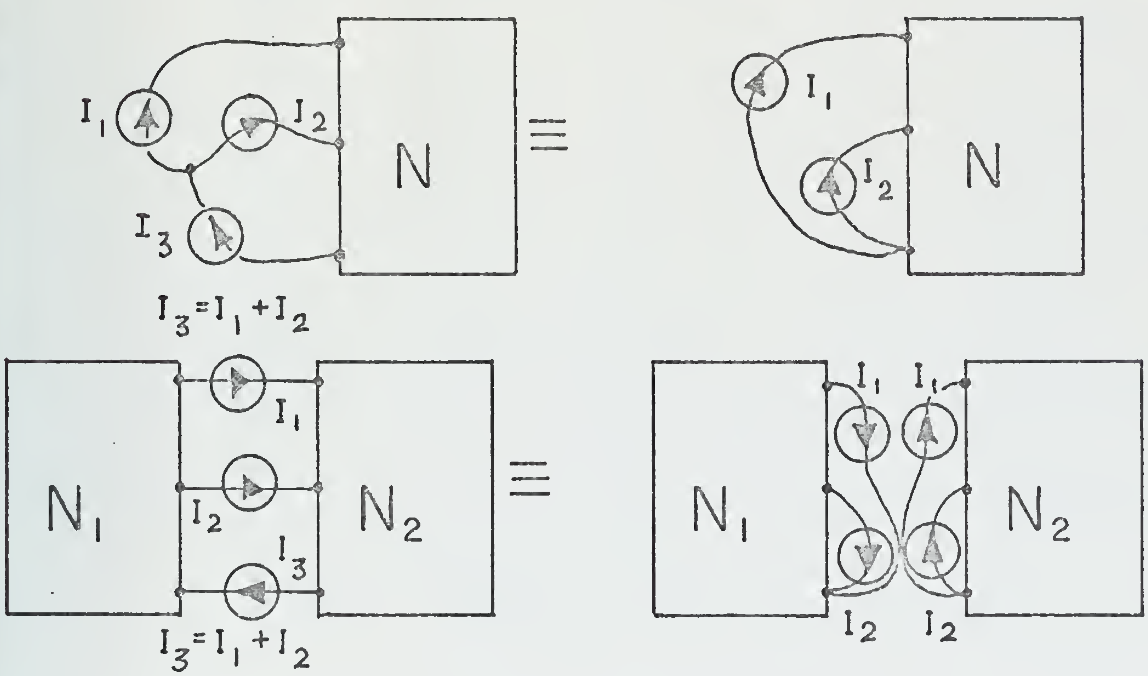
$$Q_{21} = \begin{bmatrix} r_{RL} & (\mu_{RC} - F_{RC}) \\ (h_{GL} + F'_{LG}) & g_{GC} \end{bmatrix} \; ; \quad Q_{22} = \begin{bmatrix} r_{RR} & (\mu_{RG} - F_{RG}) \\ (h_{GR} + F'_{RG}) & g_{GG} \end{bmatrix} \; ; \quad Q_{23} = \begin{bmatrix} r_{RS} & \mu_{R\Gamma} \\ h_{GS} & g_{G\Gamma} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \tag{4e}$$

$$Q_{31} = \begin{bmatrix} r_{SL} & (\mu_{SC} - F_{SC}) \\ (h_{\Gamma L} + F'_{L\Gamma}) & g_{\Gamma C} \end{bmatrix} \; ; \quad Q_{32} = \begin{bmatrix} r_{SR} & \mu_{SG} \\ h_{\Gamma R} & g_{\Gamma G} \end{bmatrix} \; ; \quad Q_{33} = \begin{bmatrix} r_{SS} & \mu_{S\Gamma} \\ h_{\Gamma S} & g_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \tag{4f}$$

$$e_x = \begin{bmatrix} e_L \\ j_C \end{bmatrix} \; ; \quad e_w = \begin{bmatrix} e_R \\ j_G \end{bmatrix} \; ; \quad e_z = \begin{bmatrix} e_S \\ j_\Gamma \end{bmatrix} \tag{4g}$$



(a) Voltage Source Loops



(b) Current Source Cut—Sets

FIG1. ELIMINATING VOLTAGE SOURCE LOOPS AND CURRENT SOURCE CUT—SETS

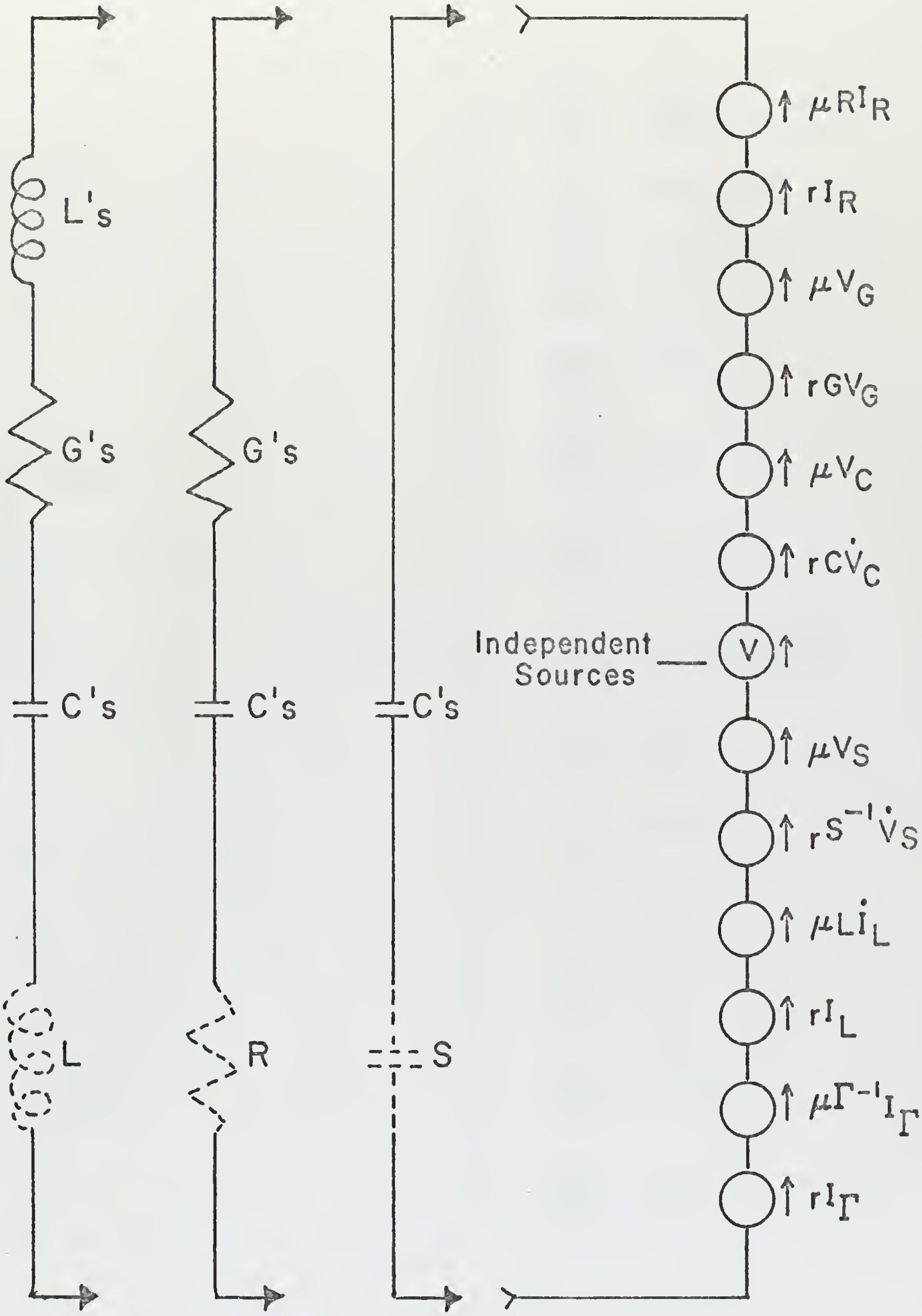


FIG 2. CONCEIVABLE LOOPS

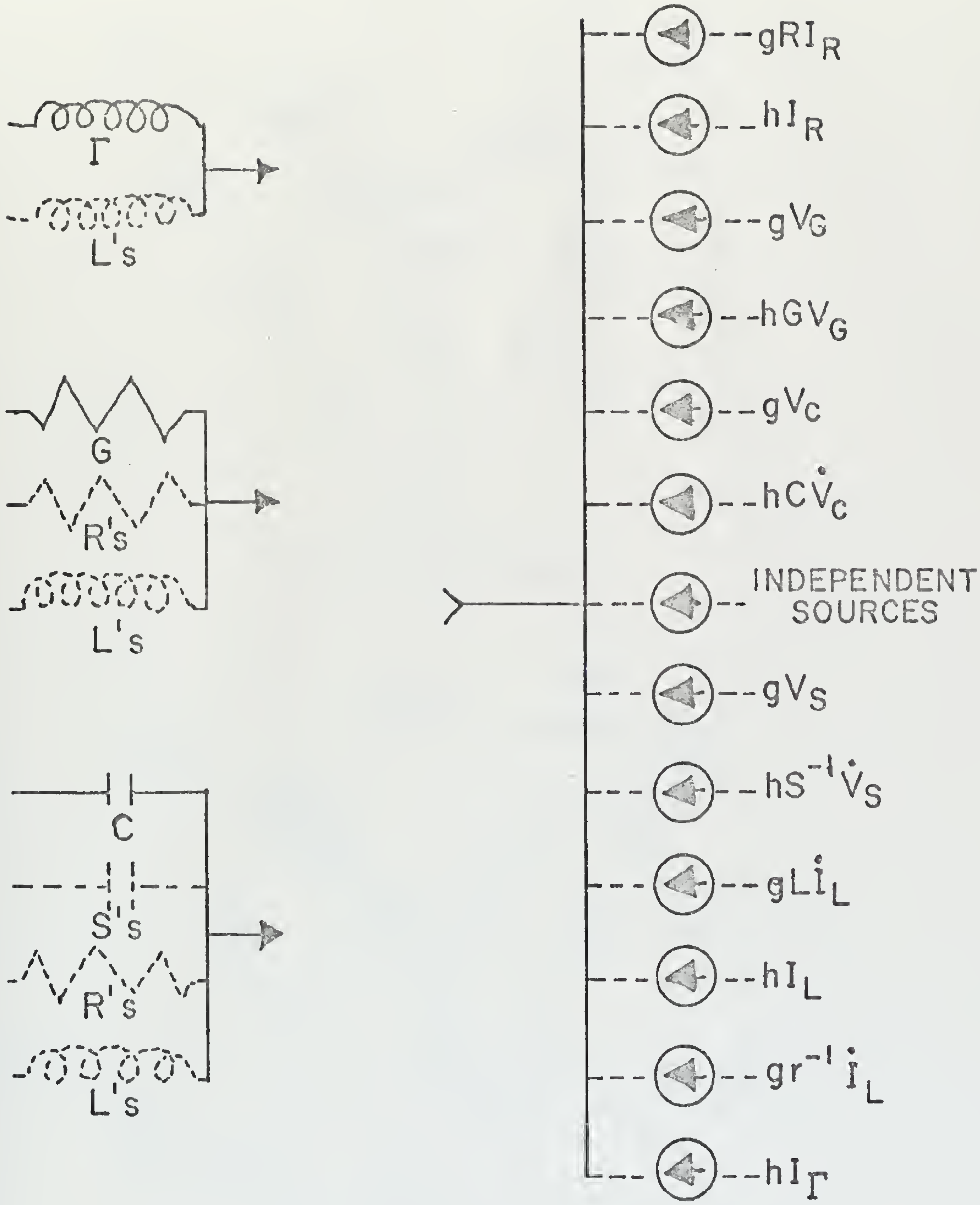
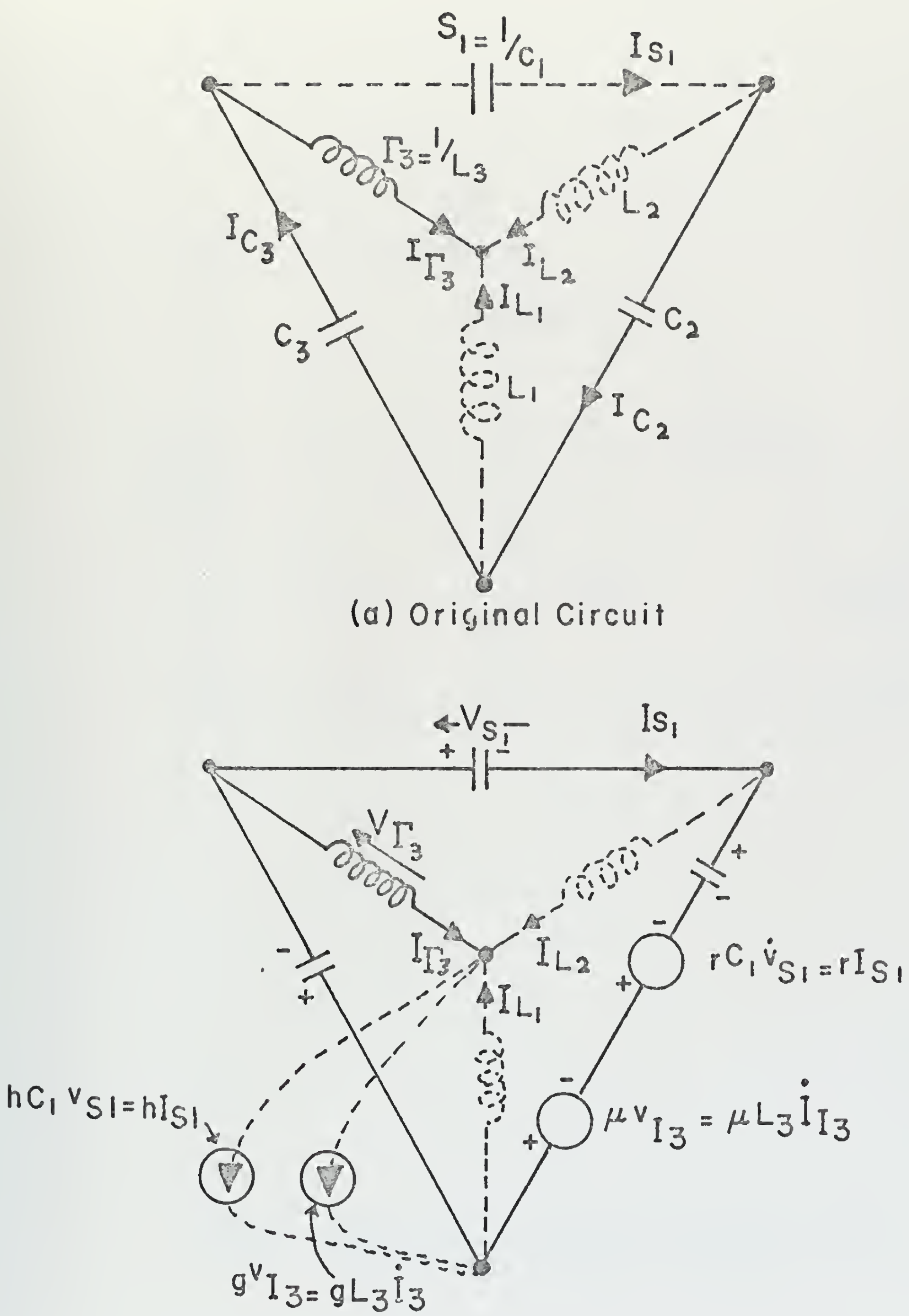


FIG 3. CONCEIVABLE CUT-SETS



(b) Circuit With Dependent Sources

FIG 4. EXAMPLE I. INTRODUCTION OF STATES BY MUTUALLY COUPLED EXCESS REACTIVE ELEMENTS

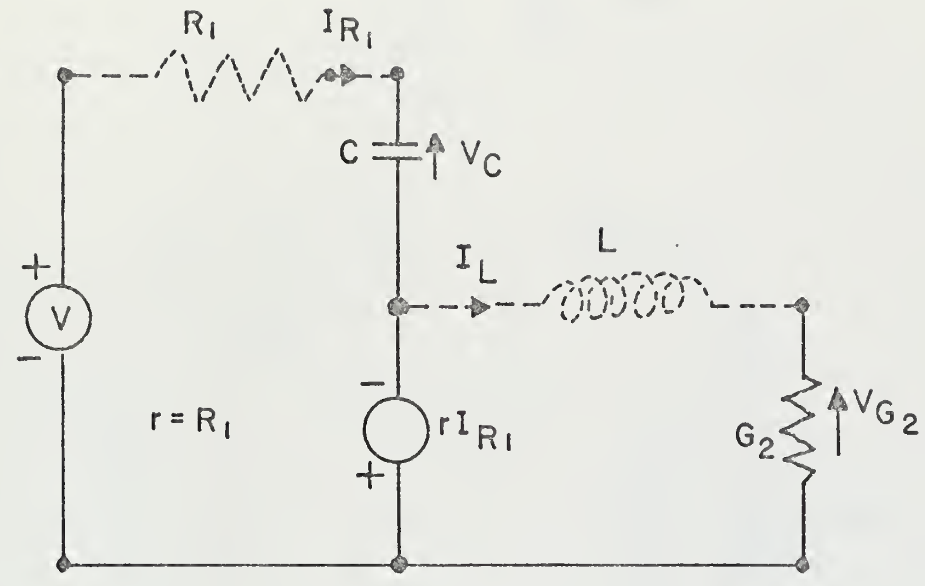


FIG 5. EXAMPLE 2. REDUCTION OF STATES DUE TO DEPENDENT SOURCE

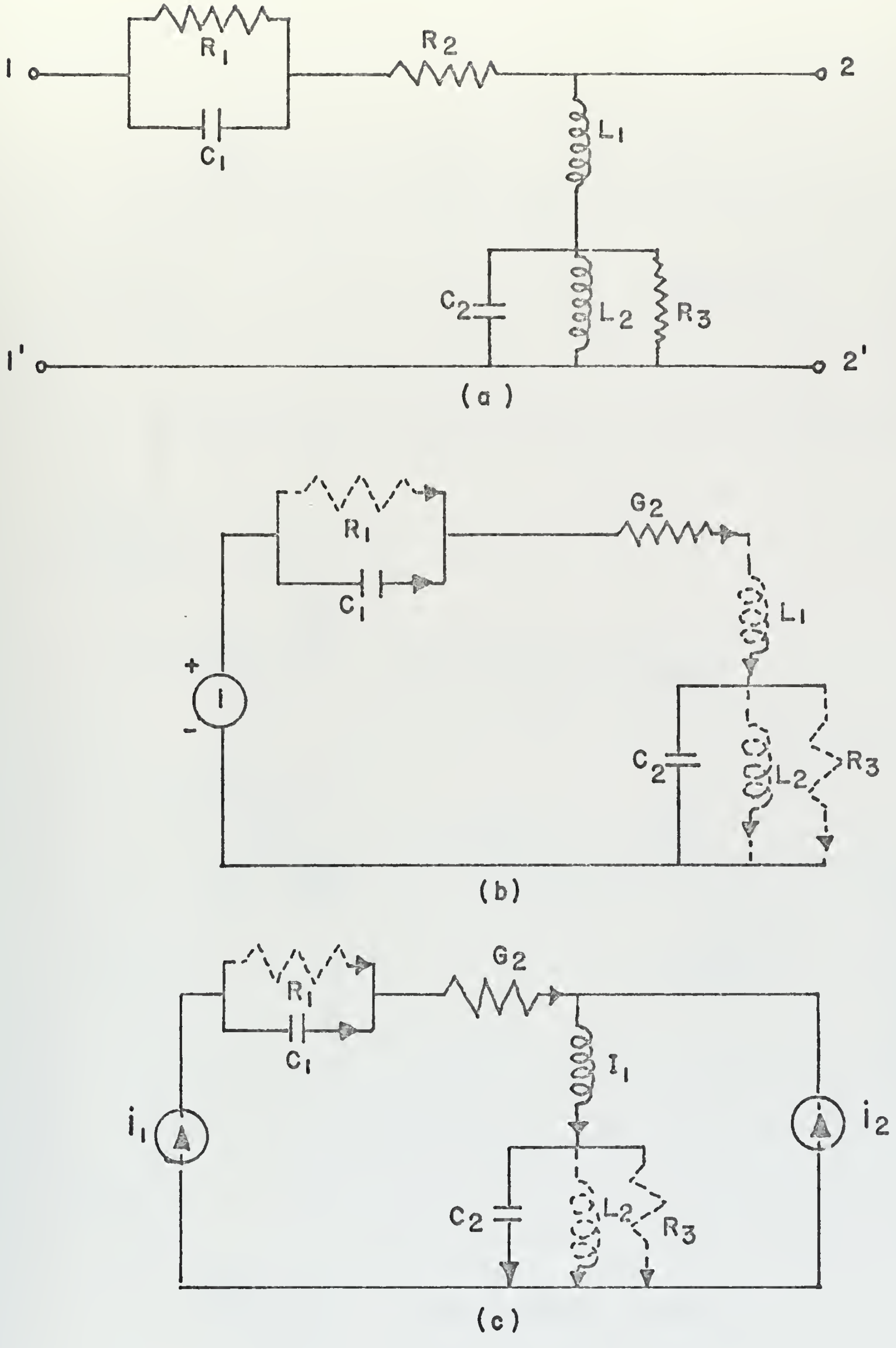
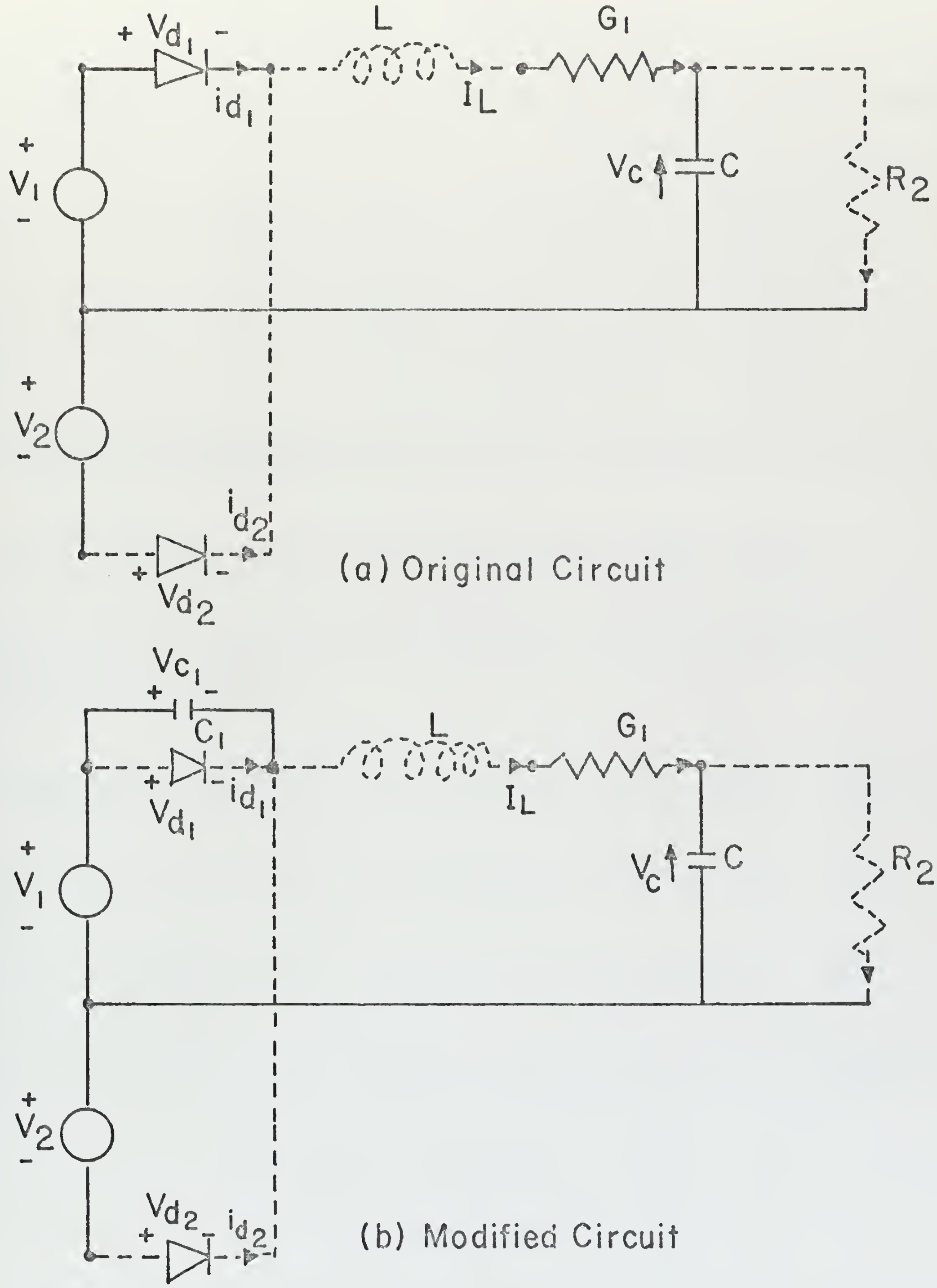


FIG 6. EXAMPLES 3 & 4. IMMITTANCE FORMS



(a) Original Circuit

(b) Modified Circuit

FIG 7. EXAMPLE 5. INTRODUCTION OF ADDITIONAL STATE TO OBTAIN NORMAL FORM REDUCTION

APPENDIX C

Elimination of Unwanted Variables

Consider the following equation and the elimination of the variable w.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} F_x(x) \\ w \\ z \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} x \\ w \\ F_z(z) \end{bmatrix} + \begin{bmatrix} e_x \\ e_w \\ e_z \end{bmatrix}$$

If $(A_{22}-B_{22})$ is nonsingular, the second equation may be written as

$$w = (A_{22}-B_{22})^{-1} \, (B_{21}x + B_{23}F_z(z) - A_{21}F_x(x) - A_{23}z + e_w)$$

Substituting this value for w into the first and third equations and combining terms yields

$$\left[\begin{array}{c|c} A_{11}-(A_{12}-B_{12})(A_{22}-B_{22})^{-1} A_{21} & A_{13}-(A_{12}-B_{12})(A_{22}-B_{22})^{-1} A_{23} \\ \hline A_{31}-(A_{32}-B_{32})(A_{22}-B_{22})^{-1} A_{21} & A_{33}-(A_{32}-B_{32})(A_{22}-B_{22})^{-1} A_{23} \end{array}\right] \begin{bmatrix} F_x(x) \\ z \end{bmatrix} =$$

$$\left[\begin{array}{c|c} B_{11}-(A_{12}-B_{12})(A_{22}-B_{22})^{-1} B_{21} & B_{13}-(A_{12}-B_{12})(A_{22}-B_{22})^{-1} B_{23} \\ \hline B_{31}-(A_{32}-B_{32})(A_{22}-B_{22})^{-1} B_{21} & B_{33}-(A_{32}-B_{32})(A_{22}-B_{22})^{-1} B_{23} \end{array}\right] \begin{bmatrix} x \\ F_z(z) \end{bmatrix}$$

$$+ \begin{bmatrix} e_x-(A_{12}-B_{12})(A_{22}-B_{22})^{-1} e_w \\ e_z-(A_{32}-B_{32})(A_{22}-B_{22})^{-1} e_w \end{bmatrix}$$

The pattern for the multiplication and combining of terms is readily established.

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BSTRACT

The reduction of circuit equations to normal form for numerical integration s considered for the general circuit where excess reactive elements, all types of nearly dependent sources, and nonlinear dissipative and reactive elements are esent. For the linear circuit, necessary and sufficient conditions for the istence of numerical solutions are considered and stated. For the nonlinear rcuit, reduction to normal form is not always possible. Numerical solution is own to be simplified if certain a priori conditions are satisfied in formulating the ginal circuit equations. A new systematic reduction procedure is presented for taining the normal form equations. This procedure is also extended to a new ocedure for obtaining transfer and immittance functions of the linear circuit from a per tree formulation.

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